Polynomial Representation for Long Knots

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Abstract

We discuss the polynomial representation for long knots and elaborate on how to obtain them with a bound on degrees of the defining polynomials, for any knot-type.

Key words: Degree Sequence, Quasitoric braids, Real deformation.

2000 Mathematics Subject Classification: Primary 57M25; Secondary 14P25.

1 Introduction

This paper is aimed to be a survey article on the polynomial representation of knots and includes all the results proved by the authors. A. Durfee and D. Oshea [1] wrote a similar paper in 2006. Our paper provides a more constructive approach and uses recent knot theoretic results. Polynomial Representation for long knots were first shown by Shastri [15]. Shastri proved that for every knot-type K ($\mathbb{R} \hookrightarrow \mathbb{R}^3$) there exist real polynomials $f(t), g(t)$ and $h(t)$ such that the map $t \mapsto (f(t), g(t), h(t))$ from $\mathbb{R}$ to $\mathbb{R}^3$ represents K and in fact the above map defines an embedding of $\mathbb{C}$ in $\mathbb{C}^3$. Shastri's motivation for proving this theorem was perhaps to find a non-rectifiable polynomial embedding of complex affine line in complex affine space which could prove one of the famous conjecture of Abhayankar [2]. However, this conjecture is still open. A knot represented by a polynomial embedding is refered as a polynomial knot.

[1] Thanks to our visit to Osaka City University in summer 2004 sponsored by COE program of Prof Akio Kawauchi where we learnt about the quasi toric braids and the Manturov's theorem.

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Similar to the case of Harmonic knots [17] and Holonomic knots [5] it has been proved that the space of all knots [14], up to ambient isotopy, can be replaced by the space of all polynomial knots, up to polynomial isotopy [16]. However polynomial parametrization of knots not only represents the knot type, it also can represent a specific knot diagram. For instance, it has been shown that we can capture the symmetric behavior of knots such as strong invertibility and strong negative amphicheirality in their polynomial representation [9]. Polynomial knots are also important from the point of view of computability as we can easily obtain three dimensional graphs of polynomial knots using *Mathematica* or *Maple* as illustrated in section 6.

In Shastri's theorem the existence of a polynomial representation for a given knot type K is shown by using Weierstrass' approximation. Thus, estimating the degrees of the polynomials is not clear. Suppose $t \mapsto (f(t), g(t), h(t))$ is a polynomial representation of a knot type K and $deg(f(t)) = l,\ deg(g(t)) = m$ and $deg(h(t)) = n$, then we say that (l, m, n) is a *degree sequence* of K. We define (l, m, n) to be the *minimal degree sequence* of K if (l, m, n) is minimal amongst all the degree sequences of K with respect to the usual lexicographic ordering of $\mathbb{N}^3$. Note that a degree sequence of K need not be unique. Also, with a given triple (l, m, n) of positive integers, the space of polynomial knots with (l, m, n) as a degree sequence is a *semi algebraic set* [4], and thus only finitely many knot-types can be realized with a given degree sequence. Hence, for a given knot-type estimating a degree sequence and eventually the minimal degree sequence is an important aspect for polynomial knots. In our earlier papers ([14],[7],[8], [10],[11], [12]) we have estimated a degree sequence and the minimal degree sequence for torus knots and 2-bridge knots.

In this paper, for any given knot type we give a concrete algorithm for writing down a polynomial representation and hence estimating a degree sequence. For this purpose, we use a recent theory of *quasi toric* braid representation of knots [6]. It is established by Manturov that every knot is ambient isotopic to some knot which is obtained by making a few crossing changes in some torus knot [6]. Thus, when we know a degree sequence for all torus knots, we can have some estimate on a degree sequence for a general knot also. Here we discuss this idea in detail and provide a method to obtain a degree sequence for a general knot type.

In section 2, we provide the basic definitions and historical background of the subject. We also include some known results without the proofs. In section 3, we state and prove the main result. We demonstrate this method and construct a polynomial representation for the knot 8_{17} in section 4. Polynomial representation of all knots up to 8 crossings and their 3d plots taken with the help of *Mathematica* are included in section 5 and section 6 respectively.

2 Some Known Results

In this section we present the basic definitions, background and the known results which serves as the prerequisite for the main result.

Definition 1 *Two non-compact knots ($\widetilde{\phi_1} : \mathbb{R} \longrightarrow \mathbb{R}^3$) and ($\widetilde{\phi_2} : \mathbb{R} \longrightarrow \mathbb{R}^3$) are said to be equivalent if their extensions ϕ_1 and ϕ_2 from S^1 to S^3 are ambient isotopic. An equivalence class of a non-compact knot is a knot-type.*

Once we have an embedding of $\mathbb{R}$ in $\mathbb{R}^3$ we may wish to see if it can be extended as an embedding of $\mathbb{C}$ in $\mathbb{C}^3$.

Remark 2 *When we have an embedding given by $t \mapsto (f(t), g(t), h(t))$ from $\mathbb{R}$ to $\mathbb{R}^3$, where $f(t)$, $g(t)$ and $h(t)$ are three real polynomials, it defines a non-compact knot and it can always be made into a polynomial embedding of $\mathbb{C}$ in $\mathbb{C}^3$ by perturbing the coefficients of any one of the polynomials.*

Definition 3 *Two Polynomial embeddings ϕ_1, $\phi_2 : \mathbb{C} \hookrightarrow \mathbb{C}^3$ are said to be algebraically equivalent if there exists a polynomial automorphism $F : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$, such that $F \circ \phi_1 = \phi_2$.*

Definition 4 *A polynomial embedding $\mathbb{C} \hookrightarrow \mathbb{C}^3$ is said to be rectifiable if it is algebraically equivalent to the standard embedding $t \mapsto (0, 0, t)$ of $\mathbb{C}$ in $\mathbb{C}^3$.*

Conjecture 5 *(Abhayankar[2]) There exist non-rectifiable embeddings of $\mathbb{C}$ in $\mathbb{C}^3$.*

The above stated conjecture is still open.

Remark 6 *If a polynomial embedding $\mathbb{R} \hookrightarrow \mathbb{R}^3$ which is also an embedding of $\mathbb{C} \hookrightarrow \mathbb{C}^3$ is rectifiable, using a polynomial automorphism with real coefficients only, it is certainly a trivial knot.*

Thus to obtain examples of non-rectifiable embeddings we must have polynomial embeddings which represent non-trivial knots. This raises the following question: *Does every knot have a polynomial representation?*

Fortunately we have an affirmative answer and the following results have been proved in this line.

Theorem 7 *[15] Every knot-type ($\mathbb{R} \hookrightarrow \mathbb{R}^3$) has a polynomial representation $t \mapsto (f(t), g(t), h(t))$ which is also an embedding of $\mathbb{C} \hookrightarrow \mathbb{C}^3$*

Theorem 8 *[16] Two polynomial embeddings ϕ_0, $\phi_1 : \mathbb{R} \hookrightarrow \mathbb{R}^3$ representing the same knot-type are polynomially isotopic. By polynomially isotopic we mean that there exists $\{P_t : \mathbb{R} \hookrightarrow \mathbb{R}^3 | \ t \in [0, 1]\}$, a one parameter family of*

polynomial embeddings, such that $P_0 = \phi_0$ *and* $P_1 = \phi_1$.

Both these theorems were proved using *Weierstrass' approximation*. Thus the nature and the degrees of the defining polynomials cannot be estimated.

Definition 9 *A triple* $(l, m, n) \in \mathbb{N}^3$ *is said to be a degree sequence of a given knot-type* K *if there exists* $f(t)$, $g(t)$ *and* $h(t)$, *real polynomials, of degrees* l, m *and* n *respectively, such that the map* $t \mapsto (f(t), g(t), h(t))$ *is an embedding which represents the knot-type* K.

Definition 10 *A degree sequence* $(l, m, n) \in \mathbb{N}^3$ *for a given knot-type is said to be the minimal degree sequence if it is minimal amongst all degree sequences for* K *with respect to the lexicographic ordering in* $\mathbb{N}^3$.

The following results have been obtained in the past.

Theorem 11 *[14] A torus knot of type* $(2, 2n + 1)$ *has a degree sequence* $(3, 4n, 4n + 1)$.

Theorem 12 *[7] A torus knot of type* $(p, q), p < q,\ p > 2$ *has a degree sequence* $(2p - 1, 2q - 1, 2q)$.

It is easy to observe that these degree sequences are not the minimal degree sequence for torus knots. For minimal degree sequence we have the following:

Theorem 13 *[8] The minimal degree sequence for torus knot of type* $(2,\ 2n+1)$ *for* $n = 3m;\ 3m+1$ *and* $3m+2$ *is* $(3,\ 2n+2,\ 2n+4)$; $(3,\ 2n+2,\ 2n+3)$ *and* $(3,\ 2n+3,\ 2n+4)$ *respectively.*

Theorem 14 *[11] The minimal degree sequence for a 2-bridge knot having minimal crossing number* N *is given by*

(1) $(3,\ N+1,\ N+2)$ *when* $N \equiv 0\ (mod\ 3)$*;*
(2) $(3,\ N+1,\ N+3)$ *when* $N \equiv 1\ (mod\ 3)$*;*
(3) $(3, N+2, N+3)$ *when* $N \equiv 2\ (mod\ 3)$

Theorem 15 *[10] The minimal degree sequence for a torus knot of type* $(p,\ 2p-1)$, $p \geq 2$ *denoted by* $K_{p,2p-1}$ *is given by* $(2p-1,\ 2p,\ d)$, *where* d *lies between* $2p+1$ *and* $4p-3$.

In order to represent a knot-type by a polynomial embedding we require a suitable *knot diagram*. For example for torus knots of type (p, q) we use its representation as closure of a p-braid namely $(\sigma_1.\sigma_2 \ldots \sigma_{p-1})^q$ and for the 2-bridge knots we use its representation as numerator closure of a *rational tangle*. For a general knot-type there may not be such systematic nice diagram available.

Definition 16 *For any two positive integers p and q, the p-braid $(\sigma_1 \dots \sigma_{p-1})^q$ is called the toric braid of type (p, q).*

Closure of a toric braid gives a torus link of type (p, q). In particular if $(p, q) = 1$, then we obtain the torus knot of type (p, q), and it is denoted by $K_{p,q}$.

Definition 17 *A braid β is said to be quasitoric of type (p, q) if it can be expressed as $\beta_1 \cdots \beta_q$, where each $\beta_j = \sigma_1^{e_{j,1}} \cdots \sigma_{p-1}^{e_{j,p-1}}$, with $e_{j,k}$ is either 1 or -1.*

A quasitoric braid of type (p, q) is a braid obtained from the standard diagram of the toric (p, q) braid by switching some of the crossing types.

Theorem 18 *(Manturov's Theorem [6]) Each knot isotopy class can be obtained as a closure of some quasitoric braid.*

3 Polynomial Representation of a general knot type

Theorem 19 *Let K be a knot which is closure of a quasitoric braid obtained from a toric braid $(\sigma_1\ \sigma_2\ \cdots\ \sigma_{p-1})^q$, where $(p, q) = 1$, by making r crossing changes. Then $(2p-1, q+r_0, d)$ is a degree sequence for K, where r_0 is the least positive integer such that $(2p-1, q+r_0) = 1$ and $d \leq 2q-1+4r$.*

In order to prove this theorem, we first prove the following lemmas.

Lemma 20 *Let $t \mapsto (f(t), g(t))$ represents a regular projection of a knot K. Let N be the number of variations in the nature of the crossings as we move along the knot. Then there exists a polynomial $h(t)$ of degree N such that the embedding $t \mapsto (f(t), g(t), h(t))$ is a representation of K.*

Proof. Let $s_1 < s_2 < \ldots < s_N$ be such that all crossings correspond to parameter values $t \in \mathbb{R} \setminus \{s_1, s_2, \ldots, s_N\}$ and in any of the open intervals $(-\infty, s_1)$, (s_1, s_2), $\ldots$, (s_N, ∞) all the crossings are of the same type (either over or under), also in successive intervals, the crossings are of opposite type. Now define

$$h(t) = \pm \prod_{i=1}^{N} (t - s_i).$$

It is easy to observe that $h(t)$ has constant sign on each interval and opposite sign on consecutive intervals, i.e. it provides an over/under crossing data for the knot-type. Hence $t \mapsto (f(t), g(t), h(t))$ is a polynomial representation of K.

Lemma 21 *Let K be a knot represented by a polynomial embedding $t \mapsto (f(t), g(t), h(t))$. Let K_r be a knot obtained from K by making r crossing changes from over to under or vice versa. Let N be the degree of $h(t)$ polynomial. Then K_r can be represented by a polynomial embedding*

$$t \mapsto (f(t), g(t), h_r(t))$$

where $deg(h_r(t))$ is at most $N + 4r$.

Proof. It can be easily shown by induction.

Proof of Theorem 19. Since the regular projection of K, is same as that of $K_{p,q}$, it has $(p-1)q$ real ordinary double points.

Case 1 *When $(q+1, 2p-1) = 1$. In this case by taking a real deformation of the curve $\tilde{C} : (t^{2p-1}, t^{q+1})$ with maximum number of real nodes, which is $(p-1)q$, we obtain a regular projection of this knot. This deformation exists by Norbart A'Campo's Theorem [3].*

Case 2 *When $(q+1, 2p-1) \neq 1$. Here we choose the least positive integer r_0 such that $(2p-1, q+r_0) = 1$ and consider the curve $\tilde{C} : (X(t), Y(t)) = (t^{2p-1}, t^{q+r_0})$. The maximum number of double points in a deformation of this curve is $(p-1)(q+r_0-1) = (p-1)q + (r_0-1)(p-1)$. By a result from real algebraic geometry [13], we can choose a real deformation $\tilde{C} : (X(t), Y(t)) = (f(t), g(t))$ with $deg(f(t)) = 2p-1$, $deg(g(t)) = q+r_0$ such that $\tilde{C}$ has $(p-1)q$ real nodes and $(r_0-1)(p-1)$ imaginary nodes.*

Observe that the crossing data for this knot differs from that of $K_{p,q}$ at r places, by Lemma 21, there exists a polynomial $\tilde{h}(t)$, with degree $d \leq 2q - 1 + 4r$, which provides an over/under crossing data for K. Thus we have shown that $(2p-1, q+r_0, d)$ is a degree sequence for K.

4 The knot 8_{17}

Consider the knot 8_{17} whose minimal braid representation is $\sigma_1^2\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-2}$. Now by using the relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_1$ we can replace $\sigma^{-1}\sigma_2$ by $\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}$ and obtain an equivalent braid representation as:

$$\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2.$$

This is a quasitoric braid representation for 8_{17}. We see that it is obtained by making crossing changes in the toric braid $(\sigma_1\sigma_2)^7$. Thus, a regular projection

of 8_{17} is same as a regular projection of a torus knot of type $(3,7)$. To obtain a regular projection we consider the parametric plane curve:

$(x(t), y(t)) = (f(t), g(t)) = (t(t^2-6.431)(t^2-15.91), t(t^2-0.18)(t^2-2.4899)(t^2-17.458)(t^2-16.15)(t^2-14.8)(t^2-11))$.

The projection is shown in figure 4.1 below

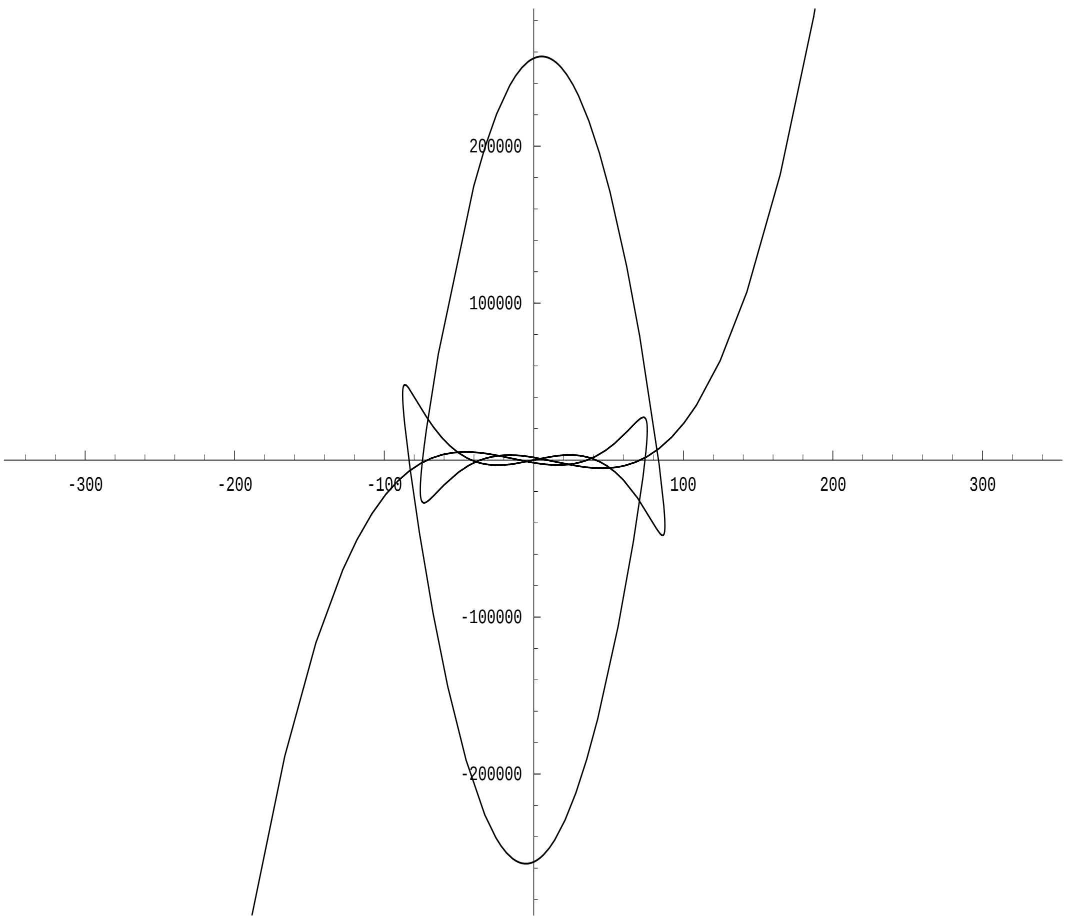

figure 4.1

Observe that 8_{17} Knot is obtained from the toric braid representation of $(3,7)$-torus with 7 crossing changes. From the quasitoric representation we notice that there are 9 variations in the signs at the crossing points as we move along the knot. Thus when we compute the parametric values at the crossing points we construct the polynomial $h(t)$ using the algorithm in Lemma 20, as $h(t) = (t+4.138362)(t+3.86)(t+2.416735)(t+1.2)(t)(t-2.416735)(t-1.2)(t-3.86)(t-4.138362)$. This polynomial provides us the under/over crossing data for 8_{17}. A 3 dimensional plot for the curve given by $(x(t), y(t), z(t)) = (f(t), g(t), h(t))$

is shown below in figure 4.2.

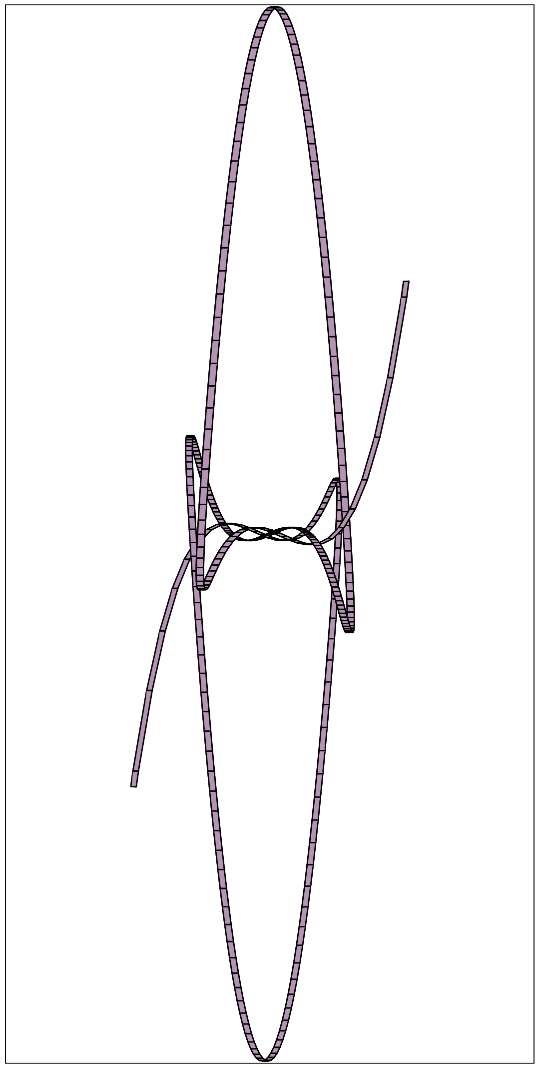

figure 4.2

5 Polynomial representation of all knots upto 8 crossings

Here we include polynomial representations for all knots upto 8 crossings. One can plot their 3 dimensional plots in *Mathematica* or *Maple* by giving the "*spacecurve*" command. 3_1 **Knot:**

$t \mapsto (t(t-1)\times(t+1), t^2(t-1.15)\times(t+1.15), (t^2-1.056445^2)\times(t^2-0.644893^2)t)$

4_1 **Knot:**

$t \mapsto (t(t-2)\times(t+2), (t-2.1)\times(t+2.1)t^3, (t^2-2.176385^2)\times(t^2-1.83588^2)\times$

$(t^2 - 0.8956385^2)t)$

$\underline{5_1 \textbf{ Knot}}$:

$t \mapsto (t^3 - 4t, (t^2 - 1.2) \times (t^2 - 2.25) \times (t^2 - 3.9) \times (t^2 - 4.85), (t^2 - 2.26311^2) \times (t^2 - 2.116775^2) \times (t^2 - 1.812575^2) \times (t^2 - 1.2655995^2)t)$

$\underline{5_2 \textbf{ Knot}}$:

$t \mapsto (t^3 - 17t, t^7 - 0.66t^6 - 29t^5 + 43t^4 + 208t^3 - 680t^2 - 731t, (t+4.5) \times (t+4.1) \times (t+3.2) \times (t+2.3) \times (t+0.85) \times (t-0.2) \times (t-1.75) \times (t-3.65) \times (t-4.59949))$

$\underline{6_1 \textbf{ Knot}}$:

$t \mapsto (-16t + t^3, (t + 3.11918) \times (t - 4.519)t(t - 1.95) \times (t + .9) \times (t + 3.85) \times (t - 1.6125) \times (t + 3.0125) \times (t + 4.35), (t + 4.53430) \times (t + 4.247655) \times (t + 3.717045) \times (t + 3.23695) \times (t + 2.257325) \times (t + 1.22656) \times (t + 0.4656315) \times (t - 0.6227735) \times (t - 1.71873) \times (t - 3.401345) \times (t - 4.51885))$

$\underline{6_2 \textbf{ Knot}}$:

$t \mapsto ((t^2 - 12) \times (t^2 - 11), t(t^2 - 21) \times (t^2 - 7), (t^2 - 4.6573875^2) \times (t^2 - 4.472939^2) \times (t^2 - 3.504525^2) \times (t^2 - 2.318071^2) \times (t^2 - 1.2983325^2)t)$

$\underline{6_3 \textbf{ Knot}}$:

$t \mapsto (t(t^2 - 16), t^7 - 2.32015t^6 - 37.8493t^5 + 68.303t^4 + 294.038t^3 - 486.111t^2 + 787.942t, (t + 4.385695) \times (t + 4.02568) \times (t + 3.51085) \times (t + 2.11637) \times (t + 1.196606) \times (t + 0.1474538) \times (t - 0.7428872) \times (t - 2.52954) \times (t - 3.810875) \times (t - 4.211535) \times (t - 4.53938))$

$\underline{7_1 \textbf{ Knot}}$:

$t \mapsto (t^3 - 3t, (t^2 - 1) \times (t^2 - 1.46) \times (t^2 - 2) \times (t^2 - 3.8) \times (t^2 - 3) \times (t^2 - 4), (t^2 - 1.99354^2) \times (t^2 - 1.962545^2) \times (t^2 - 1.83503^2) \times (t^2 - 1.56447^2) \times (t^2 - 1.293475^2) \times (t^2 - 1.09503^2)t)$

$\underline{7_2 \textbf{ Knot}}$:

$t \mapsto (-14t + t^3, -1/1000(t + 3.059911918) \times (t - 4.2519) \times (t + 1.12) \times (t - 2.185) \times (t + .24) \times (t + 4.095) \times (t - 1.19) \times (t - .65125) \times (t + 3.5) \times (t + 3.935), (t + 4.272945) \times (t + 4.13026) \times (t + 3.841115) \times (t + 3.348575) \times (t + 2.91925) \times (t + 2.126655) \times (t + 1.29503) \times (t + 0.6536465) \times (t - 0.2487545) \times (t - 1.006501) \times (t - 1.72173) \times (t - 3.18092) \times (t - 4.252285))$

$\underline{7_3 \textbf{ Knot}}$:

$t \mapsto (t^3 - 18 \times t, -(t-4) \times (t+4) \times (t-4.472139) \times (t+4.62139) \times (t^2) \times (t-3) \times (t+1.86) \times (t^2-5.0795) \times (t+4.975) \times (t+2.45) \times (t^2-14), (t+4.854055) \times (t+4.71158) \times (t+4.54806) \times (t+4.23164) \times (t+2.229356) \times (t+0.0660095) \times (t-1.2478465) \times (t-2.649115) \times (t-3.458895) \times (t-3.84593) \times (t-3.993505) \times (t-4.25932) \times (t-4.469185))$

7_4 **Knot:**

$t \mapsto (t(t^2-17), t^2(t^2-18) \times (t+4.7) \times (t^2-4.15) \times (t-4.7), (t^2-4.6^2) \times (t^2-4.35^2) \times (t^2-4.18^2) \times (t^2-9) \times (t^2-1.8^2) \times (t^2-0.75^2)t)$

7_5 **Knot:**

$t \mapsto (-22.5t^2 + t^4, -2682.4t + 658t^3 - 46.3t^5 + t^7, (t^2-(4.68844)^2) \times (t^2-(4.42719)^2) \times (t^2-(4.12207)^2) \times (t^2-(3.336625)^2) \times (t^2-(2.579105)^2) \times (t^2-(1.3191385)^2)t)$

7_6 **Knot:**

$t \mapsto ((t-3.25) \times (t+2.95) \times (t^2-18), t(t^2-6) \times (t-3.65) \times (t+3.45) \times (t^2-24), (t+4.9267) \times (t+4.83429) \times (t+4.01544) \times (t+2.77432) \times (t+2.11661) \times (t+1.77356) \times (t+0.03743) \times (t-1.894415) \times (t-2.3901) \times (t-3.17477) \times (t-4.3213) \times (t-4.89905) \times (t-4.97255))$

7_7 **Knot:**

$t \mapsto (t(t^2-16), t^2(t^2-14) \times (t+4.45) \times (t^2-4.85) \times (t-4.46), (t+4.46) \times (t+4.192) \times (t+3.89) \times (t+3.355) \times (t+1.99) \times (t+0.68) \times (t-0.00033) \times (t-0.663) \times (t-1.944) \times (t-3.369) \times (t-3.888) \times (t-4.19) \times (t-4.47))$

8_1 **Knot:**

$t \mapsto (-13.2t + t^3, -(t+3.159911918) \times (t-4.1519) \times (t+1.16812) \times (t-2.1285) \times (t+.24) \times (t+4.095) \times (t-1.019) \times (t^2-2.25) \times (t-.65125) \times (t+3.65) \times (t+3.9035), (t+4.184905) \times (t+4.12217) \times (t+3.989295) \times (t+3.741405) \times (t+3.33439) \times (t+2.84545) \times (t+2.07536) \times (t+1.229941) \times (t+0.510687) \times (t-0.259909) \times (t-0.8836655) \times (t-1.44144) \times (t-1.885795) \times (t-3.09575) \times (t-4.152015))$

8_2 **Knot:**

$t \mapsto ((t^2-17.56) \times (t), -(t-4.09) \times (t-4.0252) \times (t+2.3) \times (t+4.4954) \times (t+4.6135499) \times (t) \times (t+2.3) \times (t-3.659028) \times (t-4.7629) \times (t+4.85) \times (t-1.066352) \times (t+2.09) \times (t-2.19764829) \times (t+1.1), 6.740910^6 + 1.033929 \times 10^7 \times t - 3.1350977 \times 10^7 \times t^2 + 6.88712 \times 10^6 \times t^3 + 1.329509292 \times 10^7 \times t^4 - 2.647645 \times 10^6 \times t^5 - 2.28354726 \times 10^6 \times t^6 + 350174.87905387 \times t^7 + 200811.8844882988 \times t^8 - 22575.77543179 \times t^9 - 9585.64923687 \times t^{10} + 720.761949 \times t^{11} + 236.8726 \times$

$t^{12} - 9.13684272055 \times t^{13} - 2.378845 \times t^{14})$

8_3 Knot:

$t \mapsto (t(t^2-16), (t^2-4.58^2) \times (t^2-4.1^2) \times (t) \times (t^2-1.759^2) \times (t^2-1.98^2) \times (t^2-2^2) \times (t^2-4.13^2), -1.253835 \times 10^6 \times t + 1.89348 \times 10^6 \times t^3 - 93478 \times t^5 - 24316.5388 \times t^7 + 2306.0931 \times t^9 - 54.3551 \times t^{11})$

8_4 Knot:

$t \mapsto (-17.0275 \times t + t^3, -7238.08 \times t + 2156.21 \times t^2 + 9252.69 \times t^3 - 2762.63 \times t^4 - 2278.64 \times t^5 + 686.535 \times t^6 + 154.598 \times t^7 - 47.6818 \times t^8 - 3.15162 \times t^9 + t^{10}), -366161 + 779611.474 \times t + 1.34584 \times 10^6 \times t^2 - 3.74732 \times 10^6 \times t^3 + 1.40482 \times 10^6 \times t^4 + 2.93897 \times 10^6 \times t^5 - 991572 \times t^6 - 528142 \times t^7 + 171904.38 \times t^8 + 38975.6 \times t^9 - 12649.82 \times t^{10} - 1290.85 \times t^{11} + 425.449 \times t^{12} + 15.9088 \times t^{13} - 5.39827 \times t^{14})$

8_5 Knot:

$t \mapsto (102.6 \times t - 9.6 \times t^2 - 22.4125 \times t^3 + 0.6 \times t^4 + t^5, 913.915 - 10.1304 \times t - 1223.62 \times t^2 + 13.0047 \times t^3 + 342.138572 \times t^4 - 3.0753 \times t^5 - 33.43324 \times t^6 + 0.201 \times t^7 + t^8, -797977.96519 - 1.80074 \times 10^6 \times t + 5.20615 \times 10^6 \times t^2 - 3.14955 \times 10^6 \times t^3 - 1.8510277 \times 10^6 \times t^4 + 2.2888687 \times 10^6 \times t^5 - 2548.54 \times t^6 - 537208.72308 \times t^7 + 74673.549 \times t^8 + 57393.8 \times t^9 - 11170.2417 \times t^{10} - 2854.362 \times t^{11} + 633.4472 \times t^{12} + 53.41236 \times t^{13} - 12.658 \times t^{14})$

8_6 Knot:

$t \mapsto (21.56135 - 22.56135 \times t^2 + t^4, 57430.3 \times t^3 - 15828.94409 \times t^5 + 1554.4382 \times t^7 - 65.335787 \times t^9 + t^{11}, 3.83628 \times 10^6 \times t - 3.80231 \times 10^7 \times t^3 + 2.70379 \times 10^7 \times t^5 - 7.48485 \times 10^6 \times t^7 + 1.0666 \times 10^6 \times t^9 - 85990.4898 \times t^{11} + 3968.12 \times t^{13} - 97.833 \times t^{15} + t^{17})$

8_7 Knot:

$t \mapsto (18.1476 \times t - t^3, 334796 - 355346.8 \times t - 139874 \times t^2 + 174113.2839 \times t^3 + 26617.726 \times t^4 - 28780.456 \times t^5 - 2402.053 \times t^6 + 2153.9814214 \times t^7 + 99.996212 \times t^8 - 75.2884 \times t^9 - 1.547 \times t^{10} + t^{11}, 1.135485 \times 10^7 - 1.06477 \times 10^7 \times t - 1.61826 \times 10^7 \times t^2 + 1.597485 \times 10^7 \times t^3 + 5.23349 \times 10^6 \times t^4 - 3.97577 \times 10^6 \times t^5 - 765060 \times t^6 + 438043 \times t^7 + 59464.6 \times t^8 - 25179.9 \times t^9 - 2548.08 \times t^{10} + 741.202 \times t^{11} + 56.95 \times t^{12} - 8.8426 \times t^{13} - 0.52 \times t^{14})$

8_8 Knot:

$t \mapsto (6.392595 - 7.04755 \times t - 0.255 \times t^2 + t^3, 261.061 \times t + 63.7029 \times t^2 - 662.543 \times t^3 - 165.758 \times t^4 + 339.24453 \times t^5 + 91.9902 \times t^6 - 60.5558383355 \times t^7 - 17.4661813 \times t^8 + 3.3461 \times t^9 + t^{10}, -11243.145144 + 57682.7791 \times t - 121485 \times t^2 + 129882.5142 \times t^3 - 40537 \times t^4 - 59511.45 \times t^5 + 51887.397 \times t^6 +$

$5765.9847 \times t^7 - 15323.3 \times t^8 + 706.65613 \times t^9 + 2141.363 \times t^{10} - 149.133 \times t^{11} - 149.772 \times t^{12} + 6.641753 \times t^{13} + 4.214027 \times t^{14})$

8_9 Knot:

$t \mapsto (t^3 - 16 \times t, -223891 \times t + 117414 \times t^3 - 21544.8 \times t^5 + 1787.65 \times t^7 - 68.79768 \times t^9 + t^{11}, -6.2307944 \times 10^6 \times t + 9.3007326 \times 10^6 \times t^3 - 2.61655 \times 10^6 \times t^5 + 326427 \times t^7 - 21124.3 \times t^9 + 696.354 \times t^{11} - 9.2731 \times t^{13})$

8_{10} Knot:

$t \mapsto (6.9316 \times t - 2.38335 \times t^2 - 6.01325 \times t^3 + 0.465 \times t^4 + t^5, -7.404583 + 0.297817 \times t + 13.6921 \times t^2 - 0.307817 \times t^3 - 7.2875 \times t^4 + 0.01 \times t^5 + t^6, 56.115 - 224.5194 \times t - 466.266 \times t^2 + 609.281 \times t^3 + 1362.664 \times t^4 - 764.397 \times t^5 - 1657.83276 \times t^6 + 480.98677 \times t^7 + 992.291 \times t^8 - 146.86217 \times t^9 - 303.83859 \times t^{10} + 20.8645 \times t^{11} + 45.67642 \times t^{12} - 1.10155 \times t^{13} - 2.676137 \times t^{14})$

8_{11} Knot:

$t \mapsto (-t^3 + 18.1846 \times t, 71592.5 + 53428.5 \times t - 98454.7 \times t^2 - 72895.3 \times t^3 + 25882.5 \times t^4 + 17475.87426 \times t^5 - 2701.076 \times t^6 - 1634.637328 \times t^7 + 123.0385 \times t^8 + 66.87181 \times t^9 - 2.0261152 \times t^{10} - t^{11}, -2.6439 \times 10^6 + 9.11332 \times 10^6 \times t - 886938.328 \times t^2 - 1.29908 \times 10^7 \times t^3 + 6.06538 \times 10^6 \times t^4 + 5.47188 \times 10^6 \times t^5 - 2.228788 \times 10^6 \times t^6 - 812987.32456 \times t^7 + 305430 \times t^8 + 54586.184 \times t^9 - 19725.3164 \times t^{10} - 1709.51 \times t^{11} + 608.914 \times t^{12} + 20.35213 \times t^{13} - 7.260873 \times t^{14})$

8_{12} Knot:

$t \mapsto (t^3 - 18.1846 \times t, -47870.34 \times t + 65021.98 \times t^3 - 15853.7565 \times t^5 + 1520.73 \times t^7 - 64.260848 \times t^9 + t^{11}, -86291.29115469387 \times t + 542095.213 \times t^3 + 1.2745 \times 10^6 \times t^5 - 304536.6 \times t^7 + 26001.8 \times t^9 - 968.456472 \times t^{11} + 13.3503267 \times t^{13})$

8_{13} Knot:

$t \mapsto (t^3 - 17.0275, -6878.17 \times t + 1971.48 \times t^2 + 8832.4054 \times t^3 - 2534.74 \times t^4 - 2214.42774 \times t^5 + 638.572 \times t^6 + 157.241 \times t^7 - 45.88093 \times t^8 - 3.38257 \times t^9 + t^{10}, -167328.2 + 548139.2 \times t + 467343.298 \times t^2 - 2.56039 \times 10^6 \times t^3 + 1.531225328 \times 10^6 \times t^4 + 2.42578 \times 10^6 \times t^5 - 902477.197 \times t^6 - 470545 \times t^7 + 155821.49679 \times t^8 + 37082.845 \times t^9 - 11772.7196 \times t^{10} - 1311.79 \times t^{11} + 411.0726624818 \times t^{12} + 17.32924 \times t^{13} - 5.444827 \times t^{14})$

8_{14} Knot:

$t \mapsto (t^3 - 17.0275 \times t, 6463.5 \times t - 1753.5626153 \times t^2 - 8351.885 \times t^3 + 2267.07033 \times t^4 + 2144.774385 \times t^5 - 583.356 \times t^6 - 160.952 \times t^7 + 43.9894 \times t^8 + 3.63382 \times t^9 - t^{10}, 15567.9 + 362867.427 \times t - 706636 \times t^2 - 1.1441 \times 10^6 \times t^3 + 1.66678 \times 10^6 \times t^4 + 1.7645677 \times 10^6 \times t^5 - 759367.37167 \times t^6 - 394719 \times t^7 + 130227.6 \times$

$t^8 + 34548.7 \times t^9 - 10301.75 \times t^{10} - 1338.163646615 \times t^{11} + 381.8570624 \times t^{12} + 19.185648 \times t^{13} - 5.378 \times t^{14})$

<u>8_{15} **Knot**</u>:

$t \mapsto (250.125 \times t - 31.75 \times t^3 + t^5, -248.752 + 282.339 \times t^2 - 34.5875 \times t^4 + t^6, -188013.238816 \times t - 1.43839 \times 10^7 \times t^3 + 4.80456 \times 10^6 \times t^5 - 621463.4408 \times t^7 + 39055.2126 \times t^9 - 1194.249 \times t^{11} + 14.2236338 \times t^{13})$

<u>8_{16} **Knot**</u>:

$t \mapsto (t^5 - 32.5 \times t^3 + 261 \times t, -293.3125 + 328.875 \times t^2 - 36.5625 \times t^4 + t^6, -39013.8 \times t - 1.66705 \times 10^7 \times t^3 + 5.35937 \times 10^6 \times t^5 - 669154 \times t^7 + 40738 \times t^9 - 1212.44 \times t^{11} + 14.1333 \times t^{13})$

<u>8_{17} **Knot**</u>:

$t \mapsto (6.5876 \times t - 2.09435 \times t^2 - 5.85825 \times t^3 + 0.365 \times t^4 + t^5, -7.4045829 + 0.297817 \times t + 13.6920829 \times t^2 - 0.307817 \times t^3 - 7.2875 \times t^4 + 0.01 \times t^5 + t^6, 54.0204 - 151.887 \times t - 457.898 \times t^2 + 332.938 \times t^3 + 1357.9144 \times t^4 - 358.81513 \times t^5 - 1684.174 \times t^6 + 204.03671 \times t^7 + 1024.084 \times t^8 - 53.2616 \times t^9 - 317.8122 \times t^{10} + 5.63785 \times t^{11} + 48.35914 \times t^{12} - 0.1496988387 \times t^{13} - 2.86492 \times t^{14})$

<u>8_{18} **Knot**</u>:

$t \mapsto (t^5 - 5.5 \times t^3 + 4.5 \times t, -7.8375 + 14 \times t^2 - 7.35 \times t^4 + t^6, -127.627 \times t + 563.155 \times t^3 - 909.757 \times t^5 + 672.438 \times t^7 - 236.4233 \times t^9 + 38.943 \times t^{11} - 2.4293 \times t^{13})$

<u>8_{19} **Knot**</u>:

$t \mapsto (t^5 - 5.5 \times t^3 + 4.5 \times t, -7.8375 + 14 \times t^2 - 7.35 \times t^4 + t^6, -10.4337 \times t + 18.5762 \times t^3 - 8.13297 \times t^5 + t^7)$

<u>8_{20} **Knot**</u>:

$t \mapsto (-6.5876 \times t + 2.09435 \times t^2 + 5.85825 \times t^3 - 0.365 \times t^4 - t^5, -7.4045829 + 0.297817 \times t + 13.6920829 \times t^2 - 0.307817 \times t^3 - 7.2875 \times t^4 + 0.01 \times t^5 + t^6, -13.5807 + 53.717 \times t + 94.7779759 \times t^2 - 106.665896 \times t^3 - 102.442 \times t^4 + 76.5135 \times t^5 + 35.0952 \times t^6 - 21.957786 \times t^7 - 3.7440720177 \times t^8 + 2.139 \times t^9)$

<u>8_{21} **Knot**</u>:

$t \mapsto (-6.5876 \times t + 2.09435 \times t^2 + 5.85825 \times t^3 - 0.365 \times t^4 - t^5, -7.4045829 + 0.297817 \times t + 13.6920829 \times t^2 - 0.307817 \times t^3 - 7.2875 \times t^4 + 0.01 \times t^5 + t^6, -43.3193 - 39.3746 \times t + 120.193 \times t^2 + 80.7083 \times t^3 - 122.983 \times t^4 - 54.46748 \times t^5 + 57.5831 \times t^6 + 14.70042 \times t^7 - 12.4378257 \times t^8 - 1.3658548 \times t^9 + t^{10})$

The 3-d plots of these long knots represented by the above polynomial embeddings, can be downloaded from the following link:

http://www.iitg.ernet.in/prabhakar/myproject.htm

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